Support Sting," *Journal of Aircraft*, Vol. 2, No. 3, 1965, pp. 202–207.

[6] Bowman, J. E. and Clayden, W. A., "Boat-Tailed Afterbodies at $M = 2$ with Gas Ejection," *AIAA Journal*, Vol. 6, No. 10, Oct. 1968, pp. 2029–2030.

[7] Heyser, A., Maurer, F., and Oberdörffer, E., "Experimental Investigation on the Effect of Tail Surfaces and Angle of Attack on Base Pressure in Supersonic Flow," AGARD Conference Proceeding 10, 1966, pp. 267–290.

Electric Arc Moving at Hypersonic Speed

JAN ROSCISZEWSKI*
Air Vehicle Corporation, San Diego, Calif.

1. Introduction

EKDAHL, Kribel, and Lovberg[1] experimentally observed a rotating spoke in an MPD arc jet in argon. The velocity of the rotation corresponds to the hypersonic Mach number of around 30 based on the speed of sound at cold flow.

As is indicated at the end of this paper, calculations based on Lovberg's experimental data indicate that the arc could behave like a solid body at hypersonic speed. This is similar to the case of arcs moving at subsonic speed as indicated by some authors.[2,3]

In the present paper, using a Newtonian pressure distribution corresponding to a thin shock layer at hypersonic speed, the shape of the arc is derived. It is assumed that the electric field and temperature are constant in the arc cross section. The last assumption is justified because the plasma is fully ionized within the spoke. A high electron conductivity results in a nearly constant temperature. Indeed, recent experiments have supported this assumption, indicating measured temperature surprisingly constant and equal about 1 ev.[4] The spoke is driving an ionizing shock wave (Fig. 1). The plasma in the thin shock layer is a source of strong line radiation.

2. Derivation of the Shape of an Arc

We shall assume a uniform applied magnetic field B and a small induced field. Conservation of momentum in the x direction gives

$$\partial p/\partial x = jB = \sigma(E - U_\infty B)B \tag{1}$$

$$\partial p/\partial y = 0 \tag{2}$$

The electric conductivity is given by the Spitzer formula (MKS units)

$$\sigma = \frac{1.5 \times 10^{-2}T^{3/2}}{\ln[1.23 \times 10^7(T^{3/2}/n_e^{1/2})]} \tag{3}$$

with

$$p = knT \tag{4}$$

one obtains from Eq. (1)

$$dknT/dx = \sigma(n,T)(E - U_\infty B)B \tag{5}$$

Assuming fully ionized gas ($n = 2n_e$) within the spoke, and $T = \text{const}$, one obtains

$$kT \int \frac{dn}{\sigma(n)} = (E - U_\infty B)Bx \tag{6}$$

or taking into account relation in (3), the preceding integral can be calculated as follows:

$$\int \frac{\ln[1.23 \times 10^7(T^{3/2}/n_e^{1/2})]dn}{1.51 \times 10^{-2}T^{3/2}} = \frac{n\{\ln[1.23 \times 10^7 T^{3/2}/(\frac{1}{2}n)^{1/2}]\}}{1.51 \times 10^{-2}T^{3/2}} + C$$

Noting that $\ln(n^{1/2})$ is a slow function of n, the expression in parenthesis can be taken as constant. This is equivalent to the constant current density throughout the arc. Therefore,

$$\int \frac{dn}{\sigma(n)} = K_1 n + C$$

where

$$K_1 = \frac{\ln\{[1.23 \times 10^7 T^{3/2}/(\frac{1}{2}n)^{1/2}]\} + 1}{1.51 \times 10^{-2}T^{3/2}}$$

Taking into account that E and B are constant, one obtains from Eqs. (6) and (4)

$$p = K_2 x + C_1$$

where

$$K_2 = \frac{(E - U_\infty B)B}{K_1} \tag{7}$$

The constant C_1 can be calculated by equating the pressure from Eq. (7) with the Newtonian stagnation point pressure $p_s = \rho_\infty U_\infty^2$ at $x = x_m$ (Fig. 1).

Therefore,

$$C_1 = \rho_\infty U_\infty^2 - K_2 x_m \tag{8}$$

denoting

$$\bar{p} = \frac{p}{\rho U_\infty^2}, \quad \bar{x} = \frac{K_2 x}{\rho_\infty U_\infty^2}, \quad \bar{y} = \frac{K_2 y}{\rho_\infty U_\infty^2} \tag{9}$$

one obtains

$$\bar{p} = 1 - (\bar{x}_m - \bar{x}) \tag{10}$$

Assuming a Newtonian pressure distribution over a solid body, i.e.,

$$\bar{p} = \cos\alpha = \frac{[d\bar{y}/d\bar{x}]^2}{1 + [d\bar{y}/d\bar{x}]^2} \tag{11}$$

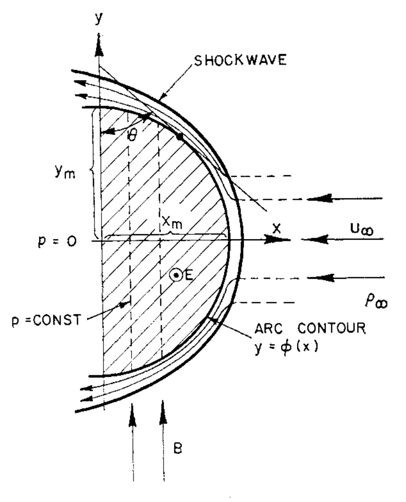

Fig. 1 Arc cross section in hypersonic flow.

Received February 13, 1969; revision received December 1, 1969. This paper has been supported by the Air Force Office of Scientific Research Contract F44620-68-C-0052. The author wishes to express his appreciation for the many fruitful discussions with R. Lovberg and R. Dethlefsen.

* Consultant.

and using Eqs. (10) and (11), one obtains

$$\frac{d\bar{y}}{d\bar{x}} = \frac{(1 - \bar{x}_m + \bar{x})^2}{1 - (1 - \bar{x}_m + \bar{x})^2}$$

Integrating, one obtains the equation of a circle

$$\bar{y}^2 + (1 + \bar{x} - \bar{x}_m)^2 = 1 \tag{12}$$

A zero value of pressure corresponds to $\bar{x} - \bar{x}_m = -1$. Of course, the current density also will be zero, and the assumption that $K_1 = \text{const}$ breaks down. However, of most importance is the front part of the arc where the pressure is relatively high.

3. Forces Acting on the Arc Column

The total dimensionless force $\bar{F}$ acting on an arc of length of 1 is

$$\bar{F} = \frac{F}{\rho_\infty U_\infty{}^2 y_m} = 2 \int_0^1 \bar{p} d\bar{y} = 2 \int_0^1 \cos\alpha d\bar{y} = \frac{4}{3}$$

The estimated gas density at the flow rate of argon (0.05 g/sec and initial axial flow velocity 600 m/sec) $\rho = 0.17 \times 10^{-4}$ kg/m^3.†

The total gas dynamic force (taking into account that for the semicircular arc $\bar{y}_m = \bar{x}_m$) is equal to

$$F_G = \tfrac{4}{3}\rho_\infty U_\infty{}^2 y_m = \tfrac{4}{3} \times 0.17 \times 10^{-4} 10^8 \times 0.02 = 46 \text{ N/m}$$

Taking experimental data of Ref. 1, $B \cong 0.1$ web/m^2, total current $I = 550$ amp, arc depth $\bar{x}_m = 0.02$ m, and arc rotation frequency $\nu = 45{,}000$ l/sec, one obtains a spoke velocity at mean radius $r \cong 0.035$ m

$$U_\infty = U = 45000 \times 2\pi \times 0.035 = 10^4 \text{ m/sec}$$

and the total electromagnetic force per unit length of the arc is

$$F_E = IB = 550 \times 0.1 = 55 \text{ N/m}$$

The agreement between F_F and F_E is in line with the accuracy of the density estimate given in Ref. 1.

4. Effect of the Centrifugal Acceleration

In the previous considerations, centrifugal acceleration was neglected, and therefore they were related to the plane hypersonic motion of an arc. Such a situation could occur in the rail gun rather than in an MPD arc jet.

With the high rotational velocity of the spoke in an MPD arc jet, the effect of a centrifugal acceleration results in the motion of plasma toward the wall.[1] The following considerations are intended to give a simple explanation of some phenomena involved.

The equation of the motion of plasma contained in the shock layer in the radial direction reads (one neglects pressure gradient which is small except near the anode)

$$\frac{dV_r}{dt} = V_r \frac{dV_r}{dr} = \frac{(U - u \cos\alpha)^2}{r} \tag{13}$$

which says that the gas in the shock layer is accelerated in a radial direction due to the centrifugal motion of the spoke. V_r denotes the radial velocity, $U - u \cos\alpha$ gas velocity in the laboratory system of coordinates.

The shock layer assumption leads to the following distribution of the average velocity u in the plane hypersonic flow in the thin shock layer around the cylinder (following from a simple one-strip integral method)

$$u/U = \tfrac{1}{2} \cos\alpha \tag{14}$$

At $\alpha = \pm 45°$ (corresponding to about 70% of the width of the spoke), the second term in Eq. (13) is 25% of the first term. Moreover, the velocity u in the present case should be smaller than in a plane due to the radial mass flow of the gas in the shock layer.

Assuming, for simplicity, that u/U is independent of r, taking into account that $U = \omega r$, where ω denotes the angular spoke velocity, integrating Eq. (13), one obtains

$$V_r = \omega r[1 - (u/U) \cos\alpha] = U[1 - (u/U) \cos\alpha] \tag{15}$$

because, as we pointed out, according to Eq. (14) the second term in parenthesis is much less than 1, we have approximately

$$V_r \cong U \tag{16}$$

and is of order of 10^4 m/sec. This large radial flow velocity in the thin shock layer preceding a spoke could result in the mass accumulation and formation of the dense gas layer at the cathode.

The thickness of some sort of the layer observed by Lovberg[1] and his collaborators is of order of 1 mm and is characterized by a large voltage drop $E_s = 4 \times 10^4$ v/m. We shall show that plasma cannot be confined in this layer but must be rather "reflected" from the wall.

If all the plasma would be deposited at the wall and flowing only in the axial direction, so that the steady flow is maintained, the density of the gas in the anode layer would be increased with respect to the density of an oncoming flow by the area ratio;

$$\varphi = r_a/2\delta \tag{17}$$

where r_a is the anode radius ($r_a{}^2 \gg r_c{}^2$, r_c being the cathode radius), δ denotes the thickness of the anode layer.

For $\delta = 10^{-3}$ m and $r_a = 0.04$ m one gets $\varphi = 20$. Therefore, density in the anode layer is

$$\rho_a = \rho\varphi = 0.17 \times 10^{-4} \times 20 = 3.4 \times 10^{-4} \text{ Kg/m}^3$$

or, corresponding number density for argon is

$$n = \frac{\rho_a}{M} = \frac{3.4 \times 10^{-4}}{1.6 \times 4 \times 10^{-26}} = 5 \times 10^{21} \frac{1}{\text{m}^3}$$

$\omega_e\tau_e$ can be calculated from the expression[5]

$$\omega_e\tau_e = \sigma B_z/n_e e \tag{18}$$

For the fully ionized gas assuming $T_e = 10^4$°K using Eq. (3) one gets

$$\sigma \cong 1.8 \times 10^3 \text{ mhos/m}$$

and

$$\omega_e\tau_e = \frac{1.8 \times 10^3 \times 0.1}{5 \times 10^{21} \times 1.6 \times 10^{-19}} = 0.23$$

Azimuthal Hall current density in an anode layer would be equal to

$$j_{e\theta} = \sigma E\omega_e\tau_e = 1.8 \times 10^3 4 \times 10^4 \times 0.23 \cong 1.66 \times 10^7 \text{ amp/m}^2$$

This current is order of magnitude larger than a radial current which is equal to

$$j_r = \frac{2I}{\pi r^2} = \frac{2 \times 550}{\pi \times 0.02^2 \text{ m}^2} = 8.75 \times 10^5 \text{ amp/m}^2$$

Azimuthal current results in a large inward pinching force

$$F_r = -j_\theta B_z = -1.66 \times 10^7 \times 0.1 = -1.66 \times 10^6 \text{N/m}^3$$

Such a force could only be balanced by a centrifugal force

$$-\rho v\theta^2/r = -j_\theta B_z$$

† The corresponding mean free path is about 2×10^{-4} m.

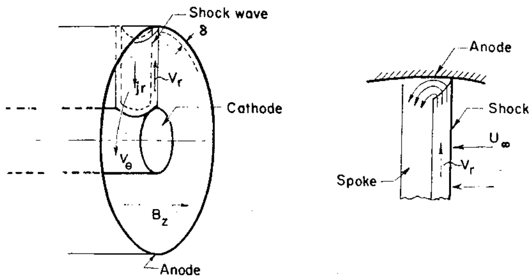

Fig. 2 Rotating arc spoke.

The corresponding azimuthal velocity is equal to

$$v_\theta = \left[\frac{r}{\rho} j_{e\theta} B_z\right]^{1/2} = \left[\frac{4 \times 10^{-2}}{3.4 \times 10^{-4}} 1.66 \times 10^6 \times 0.1\right]^{1/2} = 4.5 \times 10^3 \text{ m/sec}$$

Such a large velocity in a thin layer would result in a large friction force equal to

$$\mu \frac{\partial^2 v_\theta}{\partial r^2} \cong \mu \frac{v_\theta}{\delta^2} = 3 \times 10^{-4} \frac{4.5 \times 10^3}{(0.001)^2} = 1.35 \times 10^6 \text{ N/m}^3$$

which is order of magnitude larger than the force $j_r B_z$ driving the spoke. On this basis we should reject the model of a thin boundary layer with a spinning dense plasma.

Another possibility is that the plasma reflects from the wall and enters the spoke and moves inward (Fig. 2). This could be shown from the following consideration. The stagnation plasma pressure within the shock layer at the anode is equal to (density in the shock layer is equal to $\rho_s = (\gamma + 1)/(\gamma - 1)\rho = 4 \times 0.17 \times 10^{-4} = 0.68 \times 10^{-4}$ kg/m^3) the value calculated from the Newtonian formula

$$p_0 = \rho_s V_r^2 = 0.68 \times 10^{-4} \times 10^8 = 6.8 \times 10^3 \text{ N/m}^2$$

The pressure within the spoke is of order of

$$p_s = \rho_\infty U^2 = 0.17 \times 10^{-4} \times 10^8 = 1.7 \times 10^3$$

Because of the smaller pressure within the spoke, the pressure gradient could be larger than centrifugal force $\rho v_\theta^2/r$ and the plasma really should reflect from the wall and enter the spoke, leaving the spoke through the rear (Fig. 2).

There is experimental evidence that such a recirculation occurs[6] and the ion probe indicates that the gas is moving at the front of the spoke outward, followed by an inward motion of the plasma.

Spinning plasma entering the magnetic nozzle is further accelerated, like in the case of symmetric discharge. Energy losses should be equal to

$$m[V_r/2]^2$$

If final axial plasma velocity is equal to 10^4 m/sec, neglecting other losses, efficiency of the device with a rotating spoke would be 50%. This is an upper limit of efficiencies in this type of device.

References

[1] Ekdahl, C., Kribel, R., and Lovberg, R., "Internal Measurements of Plasma Rotation in an MPD Arc," *AIAA Bulletin*, Vol. 4, Sept. 1967, p. 432.

[2] Sabol, A. P., Stewart, R. B., and Duckett, R. J., "An Experimental Study of the Behavior of High-Current Arcs Driven by Strong External Magnetic Fields," *Proceedings of the Eighth Symposium on Engineering Aspects of Magnetohydrodynamics*, Stanford Univ., Calif., 1957, pp. 130–131.

[3] Goldstein, M. E. and Fay, J. A., "The Shape of Magnetically Balanced Arc," *AIAA Journal*, Vol. 5, No. 8, Aug. 1967, pp. 1510–1511.

[4] Lovberg, R., private communication, 1968.

[5] Sutton, G. W., ed., *Direct Energy Conversion*, McGraw-Hill, 1966, p. 188.

[6] Lovberg, R. and Kribel, R., private communication, 1968.

Effects of In-plane and Rotary Inertia on the Frequencies of Eccentrically Stiffened Cylindrical Shells

S. Parthan* and D. J. Johns†
Loughborough University of Technology, England

Accurate determination of natural frequencies of vibration of stiffened shells is essential for various engineering applications, e.g., estimation of fatigue life, supersonic flutter analysis of rockets and missiles, etc. In this Note the simple arbitrary mode defined by

$$\begin{aligned} u &= \bar{u} \cos\frac{m\pi x}{a} \cos\frac{ny}{R} \sin\omega t \\ v &= \bar{v} \sin\frac{m\pi x}{a} \sin\frac{ny}{R} \sin\omega t \\ w &= \bar{w} \sin\frac{m\pi x}{a} \cos\frac{ny}{R} \sin\omega t \end{aligned} \tag{1}$$

is used for the axial, circumferential, and radial displacements respectively in the strain and kinetic energy expressions of Refs. 1–3 to study the influence of various inertia terms on the invacuo frequencies of vibration of simply supported eccentrically stiffened circular cylindrical shells and to examine the efficacy of 1) stiffener discreteness as compared to stiffener smearing and 2) stiffener configuration.

For the discrete stiffener analyses it is assumed that the $2L$ stringers and the $(K + 1)$ rings are located at positions determined respectively by

$$\begin{aligned} y_l/R &= (2l - 1)/2L, \; l = 1,2, \ldots 2L \\ x_k/a &= k/K, \; k = 0,1,2, \ldots K \end{aligned} \tag{2}$$

This type of stiffener distribution has the advantage that their axial and radial displacements are zero when the circumferential nodes are a multiple of the number of stringers, and their circumferential and radial displacements are zero when the axial nodes are a multiple of the number of rings. Simple support boundary conditions are satisfied by this choice of stiffener distribution. The details of these analyses for the "smeared" and "discrete" stiffener cases are given in Refs. 1–2, respectively and summarized in Ref. 3.

Table 1 Properties of shells for numerical examples

	Ref. 4	Ref. 1	Ref. 2	
a, in.	40	23.75	24.00	38.85
r, in.	20	9.55	9.537	7.657
t, in.	0.04	0.028	0.0256	0.01826
E,E_s,E_r, 10^6 psi	10	10.5	10	29
ρ lb/in.^{3}	0.0998	0.095	0.0975	0.2819
ν	0.3	0.3	0.315	0.3
b_s,b_r, in.	...	0.096	0.1118	0.0409
h_s,h_r, in.	...	0.302	0.2262	0.3981
L	...	60	60	4
K	...	25	...	...
d, in.	...	1	...	...
l, in.	...	1	...	...

Presented at the AIAA Structural Dynamics and Aeroelasticity Specialist Conference, New Orleans, La., April 16–17, 1969 (no paper No.; published in bound volume of conference papers); submitted June 4, 1969; revision received November 3, 1969. Work supported by Ministry of Technology under Contract PD/170/07.

* Research Assistant.

† Professor.